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Gravi-weak unification

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Abstract
The coupling of chiral fermions to gravity makes use only of the selfdual $SU(2)$ subalgebra of the (complexified) $SO(3, 1)$ algebra. It is possible to identify the antiselfdual subalgebra with the $SU(2)_L$ isospin group that appears in the standard model, or with its right-handed counterpart $SU(2)_R$ that appears in some extensions. Based on this observation, we describe a form of unification of the gravitational and weak interactions. We also discuss models with fermions of both chiralities, the inclusion of strong interactions, and the way in which these unified models of gravitational and gauge interactions avoid conflict with the Coleman–Mandula theorem.

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1. Approaches to unification

In particle physics, the word 'unification' is used in a narrow sense to describe the following situation. One starts with two sets of phenomena described by gauge theories with gauge groups G_1 and G_2. A unified description of the two sets of phenomena is given by a gauge theory with a gauge group G containing G_1 and G_2 as commuting subgroups. In the symmetric phase of the unified theory, the two sets of phenomena are indistinguishable. The subgroups G_1 and G_2, and hence the distinction between the two sets of phenomena, are selected by the vacuum expectation value (VEV) of an 'order parameter', which is usually a multiplet of scalar fields. The standard model (SM) and its grand unified extensions work this way. On the other hand, currently popular theories that claim to provide a unification of gravity with the other interactions do not fit into this general scheme.

It is possible to unify gravity and other Yang–Mills interactions in the sense described above, if one allows the order parameter to be a set of one-forms instead of scalars [1]. In order to motivate this, let us begin by considering four one-forms $\theta^m{}_\mu$ ($m = 0, 1, 2, 3$) transforming under the 'internal' global Lorentz transformations $\theta^m{}_\mu \to S^{-1m}{}_n\theta^n{}_\mu$, which preserve the internal Minkowski metric η, and the linear coordinate transformations $\theta^m{}_\mu \to \theta^m{}_\nu\Lambda^\nu{}_\mu$. Let us suppose that the dynamics of the theory is such that θ has a constant VEV $\langle\theta^m{}_\nu\rangle = \bar\theta^m{}_\nu$ with

$$\det\bar\theta^m{}_\nu \neq 0. \tag{1}$$

This VEV breaks the original invariances but preserves the global 'diagonal' Lorentz subgroup defined by $\Lambda = \bar\theta^{-1}S\bar\theta$. Defining $\theta = \bar\theta + h$, the matrix [1] $H = \eta h\bar\theta^{-1}$ (which has two covariant Latin indices) transforms under the unbroken group as $H \to S^T H S$. Therefore, its symmetric and antisymmetric parts, which have ten and six components, respectively, are irreducible representations of this unbroken group. On the other hand, under an infinitesimal internal Lorentz transformation $S = 1 + \epsilon$, the antisymmetric part of H gets shifted: $H \to H - \eta\epsilon$. This is the typical behavior of the Goldstone bosons. As a result, if the action was invariant under the original transformations, the antisymmetric components of H would be massless.

Gravity in the first-order formulation is a gauged version of the preceding theory, where the gauge field is a local Lorentz connection $A_\mu{}^m{}_n$, the field $\theta^m{}_\mu$ is the vierbein or soldering form, defining a metric

$$g_{\mu\nu} = \theta^m{}_\mu\theta^n{}_\nu\eta_{mn}, \tag{2}$$

and its covariant curl is the torsion tensor

$$\Theta_\mu{}^m{}_\nu = \partial_\mu\theta^m{}_\nu - \partial_\nu\theta^m{}_\mu + A_\mu{}^m{}_n\theta^n{}_\nu - A_\nu{}^m{}_n\theta^n{}_\mu. \tag{3}$$

As usual in gauge theories, the Goldstone bosons can be gauged away (this corresponds to choosing the unitary gauge) and their kinetic term generates a mass for the gauge fields in the 'broken' directions. For example, if the VEV is $\bar\theta = M\mathbf{1}$, i.e. flat space, the term

$$\Theta^m_{\mu\nu}\Theta^{n\mu\nu}\eta_{mn} \tag{4}$$

generates a mass term of the form $M^2(A_{mnp} - A_{pnm})(A^{mnp} - A^{pnm})$. A similar phenomenon occurs with the Palatini action. We have here a version of the Higgs mechanism, where the six antisymmetric components of H become the six longitudinal components of the Lorentz connection. The symmetric components of H cannot be eliminated in this way: after suitably dealing with the diffeomorphism invariance, they correspond to the graviton field. It is possible to show that they are also the Goldstone bosons for a larger $GL(4)$ internal symmetry [2]; this is useful if one wants to discuss more general theories where the connection is allowed not to be metric, but this will not be needed for the purposes of this paper. The main conclusion of this discussion is that gravity is a gauge theory in the Higgs (broken) phase, and that a 'symmetric' phase would correspond to vanishing $\bar\theta$ and hence vanishing metric[4].

If one regards the soldering form as an order parameter, one sees that gravity is unification-ready. One way to achieve the unification of gravity and other gauge interactions, in the strict sense defined above, is to enlarge the internal spaces from four to N dimensions. For example, it was proposed in [1] that gravity could be unified with an $SO(10)$ GUT in the group $SO(13, 1)$, which contains the local Lorentz generators and the $SO(10)$ generators as commuting subalgebras.

In the present work, we begin to discuss this unification mechanism from the bottom up, starting from the weak interactions, rather than postulating from the outset what the unified

[4] Because in the vierbein formulation we are used to maintaining explicit invariance under the local Lorentz transformations, the statement that the local Lorentz invariance is broken spontaneously may cause some confusion. The fact is that gauge symmetries are never broken: in the Higgs phenomenon it is the choice of the unitary gauge that breaks the gauge invariance, but we are free to chose any other gauge if we want to. The physical meaning of the statement that 'the gauge symmetry is broken' is that the gauge bosons are massive; in the gravitational case, this is consistent with the fact that at low energy the connection is not an independent variable.

gauge group has to be. This means identifying the local Lorentz gauge group of gravity and the local isospin gauge group of the weak interactions as commuting subgroups in a unifying group. The direction is clearly indicated by the quantum numbers of the SM fermions. We shall see that if we consider only chiral spinors of a single handedness, it is indeed possible to achieve this goal with the group $SO(4, \mathbb{C})$. Since this group is not simple, one may object that this is not a true unification. However, we can define the representations and the action in such a way that the theory is invariant under a discrete $\mathbb{Z}_2$ group that interchanges the weak and gravitational interactions. In this sense one has a symmetric treatment of these interactions, ready for extension to truly unified groups. The manifest asymmetry that we see in the real world can be attributed to a single symmetry breaking phenomenon, occurring near the Planck scale, which also gives mass to the gravitational connection.

There are at least two different ways in which this scheme can be enlarged to account for massive fermions. One of them is based on the notion of 'algebraic spinors' and Clifford algebras. In this case, the unifying group of the gravitational and weak interactions is $GL(4, \mathbb{C})$. This approach is developed in [20]. In the other case the unifying group is $SO(7, \mathbb{C})$, and further unification with the strong interactions leads to the group $SO(13, \mathbb{C})$. This is closer to the original work in [1].

The approach to unification described in this work fits in the general framework of 'gauge theories of gravity', where the connection on spacetime plays a more prominent role than in ordinary general relativity. We refer to [6] for an extensive review of older work in this direction. Although we shall not use it explicitly, the reformulation of general relativity in terms of Ashtekar variables, which emphasizes the role of the selfdual part of the connection, is close in spirit to this work. We refer to [4] for a previous look at the standard model from this point of view. In addition to [1], the symmetry breaking aspects of gravity have also been studied in [7, 8]. In the recent literature, one also finds reference to a different form of the gravitational Higgs phenomenon, where diffeomorphisms are broken (rather than local frame rotations) and the graviton is given a mass (rather than the connection) [9, 10, 19]. This alternative Higgs phenomenon is not related to unification.

This paper is organized as follows. In section 2 we will discuss a simplified world where only fermions of a given chirality are present. If the fermions are left-handed, they couple only to the selfdual part of the Lorentz connection. We will then identify the antiselfdual component of the connection with the weak gauge fields. In section 3 we write an action functional for the fermions that make sense also in a symmetric phase, and we show how it reduces to the familiar form in the 'broken' phase. The same is done for the gauge and gravitational degrees of freedom in section 4. In section 5 we discuss more realistic extensions where fermions of both chiralities are present, and in section 6 we briefly discuss the resulting scenarios for including strong interactions. In section 7 we discuss the status of the global Lorentz invariance and the way in which the conflict with the Coleman–Mandula theorem [3] is avoided. Section 8 contains concluding comments.

2. A simple chiral world

The most striking property of the SM is that all fermions are chiral (Weyl) spinors with respect to the Lorentz transformations and either singlets or Weyl spinors with respect to the weak gauge group $SU(2)_L$. In this section, we shall consider a simplified world where all fermions are massless and the weak singlets are absent. Furthermore, we ignore strong interactions and consider only one weak (left) doublet

$$\psi_L = (\nu_L \quad e_L).$$

All other doublets can be treated in the same way.

The central observation is as follows. Because these fields are complex, they automatically carry a representation of the *complexified* Lorentz and weak groups. The algebra of the complexified Lorentz group $SO(3,1,\mathbb{C})$ consists of real linear combinations of the rotation generators L_j, the boost generators K_j and their purely imaginary counterparts $\mathrm{i}L_j$ and $\mathrm{i}K_j$. In the case of the chiral fermion fields, the physical rotations and boosts are realized by the generators $M_j^+ = L_j + \mathrm{i}K_j$ and $\mathrm{i}M_j^+$, respectively, which together generate a group $SL(2,\mathbb{C})_+$. The generators $M_j^- = L_j - \mathrm{i}K_j$ of $SO(3,1,\mathbb{C})$ commute with M_j^+ and can therefore be identified with physical operations on spinors that have nothing to do with the Lorentz transformations. In our simplified model, we will identify $SL(2,\mathbb{C})_+$ with the Lorentz group, and the group generated by the M_j^- with the weak isospin gauge group $SU(2)_L$. The generators $\mathrm{i}M_j^-$ are related to the weak isospin generators in the same way as the boosts are related to the rotations, therefore we will call them 'isoboosts', and we will call the group $SL(2,\mathbb{C})_-$ generated by M_j^- and $\mathrm{i}M_j^-$ the 'isolorentz group'. It is just the complexification of the isospin group. The isoboosts are not symmetries of the world and we will discuss their fate later on. The whole group $SO(3,1,\mathbb{C}) \equiv SO(4,\mathbb{C}) = SL(2,\mathbb{C})_+ \times SL(2,\mathbb{C})_-$, which contains both the Lorentz and isolorentz transformations, will be called the 'gravi-weak' group.

To make this more explicit, we can arrange the components of ψ_L as a 2×2 matrix whose columns are (left) chiral spinors under the Lorentz group and whose rows are chiral spinors under the weak group:

$$\psi_L = \begin{pmatrix} \nu_L^1 & e_L^1 \\ \nu_L^2 & e_L^2 \end{pmatrix}. \tag{5}$$

The Lorentz group acts on this matrix by multiplication from the left and the isolorentz group acts by transposed multiplication from the right (note that this is called mathematically a left action).

The field ψ_L is therefore a bispinor: it carries a bi-index $(A\alpha)$, the first acted upon by the Lorentz and the second by the isolorentz transformations. It can also be seen as a vector of the gravi-weak group,

$$\psi_L{}^a = \hat{\sigma}^a_{A\alpha} \psi_L{}^{A\alpha}, \qquad \psi_L{}^{A\alpha} = \hat{\sigma}_a^{A\alpha} \psi_L{}^a, \tag{6}$$

where $a = 1, 2, 3, 4$, $\hat{\sigma}_a^{A\alpha}$ are the van der Waerden symbols: $\hat{\sigma}_j = \sigma_j$ $(j = 1, 2, 3)$ are the Pauli matrices and $\hat{\sigma}_4 = \mathbf{1}_4$. The matrices $\hat{\sigma}^a_{A\alpha}$ are their 'inverses'.

In this notation, the Lorentz and isolorentz groups act on ψ_L with the following generators:

$$\textit{Lorentz}: \qquad M_j^+ = \sigma_j \otimes \mathbf{1}_2 \equiv \sigma_j{}^A{}_B \delta^\alpha_\beta, \tag{7}$$

$$\textit{Isolorentz}: \qquad M_j^- = \mathbf{1}_2 \otimes \sigma_j \equiv \delta^A_B \sigma_j{}^\alpha{}_\beta. \tag{8}$$

While this terminology may sound unfamiliar, all that we have described are the standard transformation properties of a massless fermion doublet. However, reformulating things in this way is suggestive of a form of unification of the gravitational and weak interactions, with the gravi-weak group as unifying group. The gravi-weak group is the direct product of the Lorentz and isolorentz transformations, and therefore it may seem that no true unification has been achieved in this way. However, it is both mathematically and physically different to have a gauge theory of the group $SO(4,\mathbb{C})$, with a single coupling constant, and of the group $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$, which in general has two. In the following, we will investigate the possible existence of a symmetric phase of the theory where the gravi-weak invariance is manifest, and of a Higgs phase where it is broken. The world as we know it will obviously have to be identified with the latter.

We conclude this section observing that the preceding discussion can be repeated word by word inverting the roles of left and right. One would then have a simplified chiral world where the only matter is represented by massless, right-handed leptons:

$$\Psi_R = (\nu_R \quad e_R).$$

In this case, the group $SO(3, 1, \mathbb{C})$ contains the Lorentz transformations, generated by M_j^- and $\mathrm{i}M_j^-$, and the isolorentz group contains the right-weak gauge transformations M_j^+. Until we address massive fermions, in the following two sections we will continue discussing the case of left fermions only.

3. The fermionic action

To motivate what follows, we first recall the way in which one writes the action of chiral fermions coupled to gravity (not unified with other gauge fields), allowing for possibly degenerate soldering form $\theta^m{}_\mu$. We begin by noting that the soldering form carries the fundamental (vector) representation of the local Lorentz group, which is isomorphic to the tensor product of a spinor and a conjugate spinor representation. We shall use indices $A, B, \ldots$ for the spinor representation and primed indices $A', B', \ldots$ for the conjugate representation. The isomorphism is given by the van der Waerden symbols $\hat\sigma_m^{A'A}$. It is sometimes convenient to think of the soldering form as the bispinor-valued one-form

$$\theta^{AA'} = \theta^{AA'}{}_\mu \mathrm{d}x^\mu = \hat\sigma_m^{AA'}\theta^m{}_\mu \mathrm{d}x^\mu. \tag{9}$$

The fermion action must contain a spinor, a conjugate spinor and one derivative; the soldering form is the right object to covariantly contract the indices carried by these objects. The fermionic kinetic term is written as

$$\int \mathrm{d}^4x\,|\theta|\,\psi^{*A'}\theta^\mu_{A'A} D_\mu \psi^A. \tag{10}$$

Here, $D_\mu\psi^A = \partial_\mu\psi^A + \omega_\mu{}^A{}_B\psi^B$ is the Lorentz covariant derivative, $\theta^\mu_{A'A}$ is the inverse soldering form and $|\theta|$ is its determinant.

In order to have an action that makes sense also for degenerate soldering form, one writes

$$\int \psi^{*A'} D\psi^A \wedge \theta^{B'B} \wedge \theta^{C'C} \wedge \theta^{D'D} \epsilon_{(A'A)(B'B)(C'C)(D'D)}, \tag{11}$$

where

$$\epsilon_{(A'A)(B'B)(C'C)(D'D)} = \hat\sigma^m_{A'A}\hat\sigma^n_{B'B}\hat\sigma^r_{C'C}\hat\sigma^s_{D'D}\epsilon_{mnrs}$$

is antisymmetric in the exchanges of the couples of indices $(A'A)$, $(B'B)$, etc. Assuming that θ is nondegenerate, equation (11) reduces to (10).

We want to generalize the previous discussion to write an action for fermion fields coupled to weak gauge fields and gravity, in a way that is invariant under gravi-weak transformations. In the previous section we learned that the fermions can be represented as gravi-weak vectors ψ^a and their conjugates $\psi^{*\bar a}$, while the gravi-weak gauge field, in the representation carried by the fermions, is $A_\mu{}^a{}_b$. Since the gravi-weak group is gauged, we will need first of all the corresponding covariant derivative

$$D_\mu\psi_L^a = \partial_\mu\psi_L^a + A_\mu{}^a{}_b\psi_L^b. \tag{12}$$

The gauge field $A_\mu{}^a{}_b$ can be decomposed in gravitational (selfdual) and weak (antiselfdual) parts with generators (7) and (8). The (complex) gravitational gauge field will be denoted ω_μ^j, while the antiselfdual gauge fields can be further split in real part, the isospin gauge field W^j_μ, and imaginary part, the isoboosts gauge field denoted K^j_μ:

$$D_\mu \psi_L = \left[\partial_\mu + \omega^j_\mu M^{(+)}_j + \left(W^j_\mu + \mathrm{i}K^j_\mu\right)M^{(-)}_j\right]\psi_L. \qquad (13)$$

We must then define the variable describing the gravitational field. Following the argument that leads to (10), we need a generalized soldering form $\theta^{\bar{a}b} = \theta^{\bar{a}b}{}_\mu \mathrm{d}x^\mu$ whose components can be used to invariantly contract the fermion bilinear ${\psi_L}^{*\bar{a}}{\psi_L}^b$ to the covariant derivative (13).

Before proceeding further, we shall introduce a little more notation. We will use indices $m, n, \ldots$ for the vector representation of the (Lorentz) group $SL(2,\mathbb{C})_+$ and $u, v, \ldots$ for the vector representation of the (isolorentz) group $SL(2,\mathbb{C})_-$. These representations are again connected to the corresponding spinor representations by the van der Waerden symbols:

$$V^n \hat{\sigma}^{A'A}_n = V^{A'A}; \qquad V^u \hat{\sigma}^{\alpha'\alpha}_u = V^{\alpha'\alpha}. \qquad (14)$$

Using equation (6) we can convert the pair of indices $\bar{a}a$ of the generalized soldering form to a pair of bi-spinor indices: $\theta^{A'\alpha'A\alpha} = \hat{\sigma}^{A'\alpha'}_{\bar{a}} \hat{\sigma}^{A\alpha}_a \theta^{\bar{a}a}$. We can then transform the pair of indices $A'A$ to a Lorentz vector index and the pair $\alpha'\alpha$ to an isolorentz vector index. In this way, the generalized soldering form can also be written as a one-form with a Lorentz and an isolorentz vector index: $\theta^{mu} = \hat{\sigma}^m_{A'A}\hat{\sigma}^u_{\alpha'\alpha}\theta^{A'\alpha'A\alpha}$.

We can now generalize (11) and write a fermion kinetic term as

$$S_\psi = \int {\psi_L}^{*\bar{a}} D\psi^a_L \wedge \theta^{\bar{b}b} \wedge \theta^{\bar{c}c} \wedge \theta^{\bar{d}d} \epsilon_{(\bar{a}a)(\bar{b}b)(\bar{c}c)(\bar{d}d)}, \qquad (15)$$

where $\epsilon_{(\bar{a}a)(\bar{b}b)(\bar{c}c)(\bar{d}d)}$ is an $SO(4,\mathbb{C})$ invariant tensor, totally antisymmetric under interchanges of pairs of indices $(\bar{a}a)$, $(\bar{b}b)$, $(\bar{c}c)$, $(\bar{d}d)$. As we did above with θ^{mu}_μ, we can convert all the indices on this tensor to pairs of vector indices. In this notation, we chose it as follows:

$$\epsilon_{(mu)(nv)(rw)(sz)} = \epsilon_{mnrs}(\eta_{uv}\eta_{wz} + \eta_{uw}\eta_{vz} + \eta_{uz}\eta_{vw}) + (\eta_{mn}\eta_{rs} + \eta_{mr}\eta_{ns} + \eta_{ms}\eta_{nr})\epsilon_{uvwz}. \qquad (16)$$

If the combinations in the two lines on the rhs had arbitrary coefficients, this would still have the desired invariance and antisymmetry properties; the particular combination (16) is the only one that is in addition invariant under the $\mathbb{Z}_2$ group exchanging the gravitational and weak sectors.

The fermionic action (15) is built without using a metric, and therefore makes sense also in the symmetric phase. As long as $\theta^{\bar{a}a}$ has zero VEV, it does not describe a standard propagating (gaussian) theory, because it only contains interaction terms. The action is invariant under diffeomorphisms $x'(x)$ and gravi-weak gauge transformations $S^a{}_b$, under which the fields transform as

$$\psi_L \to S\psi_L, \qquad \theta_{L\mu} \to \Lambda^\nu{}_\mu S^\dagger \theta_{L\nu} S, \qquad (17)$$

where $\Lambda^\nu{}_\mu = \frac{\mathrm{d}x^\nu}{\mathrm{d}x'^\mu}$ and the gravi-weak indices have been suppressed.

The ordinary low-energy world is described by a background geometry that corresponds to Minkowski space and the flat Lorentz and isolorentz connections. Both the Lorentz and isoboost invariances must be broken at some high scale, because neither of these gauge fields appears in the low-energy spectrum. In the broken phase of the theory, at energies above the electroweak scale, the fermions can be treated as massless fields. The background geometry must thus select a timelike direction in the vector representations of the isolorentz algebra, while providing a soldering of the vector representation of the Lorentz algebra to the tangent spaces of spacetime. Selecting the timelike isolorentz direction along the fourth axis, the VEV corresponds to the choice $\langle\theta^{m4}_\mu\rangle = M\delta^m_\mu$ and $\langle\theta^{mu}_\mu\rangle = 0$ for $u = 1, 2, 3$, where M is a mass parameter. Translating back to the bispinor indices, this corresponds to

$\langle\theta_\mu\rangle = M\delta_\mu{}^m(\hat{\sigma}_m\otimes\mathbf{1}_2)$. In order to describe in a covariant fashion also the non-flat geometries with weak curvature, we will consider backgrounds of the form

$$\langle\theta_\mu\rangle = Me^m{}_\mu(\hat{\sigma}_m\otimes\mathbf{1}_2). \tag{18}$$

The dimensionless fields $e_\mu{}^m$ are now ordinary, real vierbeins connecting the tangent space index μ to the internal vector index m. Moreover, using the $SO(4,\mathbb{C})$ invariant product δ_{ab}, one can define a metric $g_{\mu\nu} = \theta_\mu^{\bar{a}a}\theta_\nu^{\bar{b}b}\delta_{ab}\delta_{\bar{a}\bar{b}}$, that using the VEV (18) is determined by the vierbeins as in equation (2):

$$g_{\mu\nu} = e^m_\mu e^n_\nu \eta_{mn}. \tag{19}$$

Note that the VEV (18) has selected $SL(2,\mathbb{C})_+$ for soldering with the spacetime transformations, and accordingly the signature of the resulting metric is minkowskian.

This VEV breaks the original group in the correct way to provide the global Lorentz and local weak (isospin) gauge invariance: the (+) part of the $SO(4,\mathbb{C})$, corresponding to the Lorentz generators (7), and the imaginary part of the (−) generators (the isoboosts) do not leave (18) invariant, and therefore are broken. Thus, the only unbroken subgroup of the original gauge group is the weak $SU(2)_L$. In addition, the special VEV $\theta_\mu^m = \delta_\mu^m$ is invariant under the global diagonal $SO(3,1)$ defined by

$$S = \mathcal{D}^{(\frac{1}{2},0)}(\Lambda), \tag{20}$$

where S and Λ are as in (17). This is the usual Lorentz group and we shall discuss its role in section 7.

At low energy the massive degrees of freedom can be ignored and the covariant derivative (13) reduces to

$$\mathcal{D}_\mu\psi_L = \big(\partial_\mu + \omega_\mu{}^i(\sigma_i\otimes\mathbf{1}_2) + W_\mu{}^i(\mathbf{1}_2\otimes\sigma_i)\big)\psi_L, \tag{21}$$

where now ω is the VEV of the spin connection constructed from $e^m{}_\mu$. It vanishes on a flat geometry, while in curved space it coincides with the Levi-Civita connection in selfdual language. Apart from strong interactions, this is the covariant derivative of left-handed fermions in the SM coupled to gravity [4]. Correspondingly, when we insert the VEV (18) in the fermionic action (15), this produces the ordinary kinetic terms for an $SU(2)_L$ doublet of canonically normalized spinors $\Psi_L = M^{3/2}\psi_L$:

$$\mathcal{S}_\psi = \int \Psi_L^{*A'\alpha'}\mathcal{D}\Psi_L^{A\alpha}\hat{\sigma}^m_{A'A}\delta_{\alpha'\alpha}\wedge e^n\wedge e^r\wedge e^s\epsilon_{mnrs} = \int \mathrm{d}^4x|e|e^\mu_m\Psi_L^{*\alpha}\hat{\sigma}^m\mathcal{D}_\mu\Psi_L^\alpha,$$

where we have suppressed the $SL(2,\mathbb{C})$ Lorentz indices in the last expression. Note the emergence of the $SU(2)$ metric $\delta_{\alpha'\alpha}$ from the antisymmetric tensor (16).

4. Gauge and gravity dynamics

We now describe the dynamics of the gauge and gravity degrees of freedom. As already mentioned, among the fluctuations, the Lorentz and isoboost gauge fields should have high masses, as they are not observed. The isospin gauge fields on the other hand should be effectively massless (until one introduces the mechanism breaking the weak interactions) and in addition, there must be a massless graviton. These degrees of freedom must emerge from a gravi-weak-invariant action in the broken phase. The action should be written in the absence of a metric, and this can be done in the first-order formalism [5]. It turns out that the possible terms that one may write are quite constrained by the gravi-weak symmetry.

The action will involve the gravi-weak-covariant combinations of derivatives of θ and A: the generalized torsion two-form

$$\Theta^{\bar{a}a}_{\mu\nu} = \partial_\mu\theta^{\bar{a}a}{}_\nu - \partial_\nu\theta^{\bar{a}a}{}_\mu + \bar{A}_\mu{}^{\bar{a}}{}_{\bar{b}}\theta^{\bar{b}a}{}_\nu + A_\mu{}^a{}_b\theta^{\bar{a}b}{}_\nu - \bar{A}_\nu{}^{\bar{a}}{}_{\bar{b}}\theta^{\bar{b}a}{}_\mu - A_\nu{}^a{}_b\theta^{\bar{a}b}{}_\mu \tag{22}$$

and the curvature two-form

$$R_{\mu\nu}{}^{\bar{a}a\bar{b}b} = R^{ab}_{\mu\nu}\delta^{\bar{a}\bar{b}} + \bar{R}^{\bar{a}\bar{b}}_{\mu\nu}\delta^{ab} \tag{23}$$

$$R_{\mu\nu}{}^{a}{}_{b} = \partial_\mu A_\nu{}^a{}_b - \partial_\nu A_\mu{}^a{}_b + A_\mu{}^a{}_c A_\nu{}^c{}_b - A_\nu{}^a{}_c A_\mu{}^c{}_b. \tag{24}$$

First, we discuss the generalized Palatini action, which contains terms linear in curvature and terms quadratic in torsion:

$$\mathcal{S}_{R1} = \frac{g_1}{16\pi}\int R^{\bar{a}a\bar{b}b}\wedge\theta^{\bar{c}c}\wedge\theta^{\bar{d}d}\epsilon_{(\bar{a}a)(\bar{b}b)(\bar{c}c)(\bar{d}d)} \tag{25}$$

$$\mathcal{S}_\Theta = a_1\int\left[t^{\bar{a}a\bar{b}b}_{\bar{e}e}\Theta^{\bar{e}e} + (t^2)\theta^{\bar{a}a}\wedge\theta^{\bar{b}b}\right]\wedge\theta^{\bar{c}c}\wedge\theta^{\bar{d}d}\epsilon_{(\bar{a}a)(\bar{b}b)(\bar{c}c)(\bar{d}d)}, \tag{26}$$

where $t^{\bar{a}a\bar{b}b}_{\bar{e}e}$ are zero-form auxiliary fields reproducing the components of $\Theta^{\bar{e}e}$.

In deriving the equations of motion (EOMs), it is convenient to split the connection and curvature in the selfdual and antiselfdual parts, converting the gravi-weak indices $(\bar{a}a)$ to the Lorentz and isolorentz indices (mu). Then, the EOMs for the isolorentz (antiselfdual) connection are identically satisfied when one inserts the VEV (18), while the equation for the Lorentz (selfdual) connection imply that the standard gravitational torsion vanishes:

$$\Theta^m_{\mu\nu} \equiv \partial_\mu e^m{}_\nu - \partial_\nu e^m{}_\mu + \omega_\mu{}^m{}_n e^n{}_\nu + \omega_\mu{}^m{}_n e^n{}_\mu = 0. \tag{27}$$

This fixes $\omega_\mu{}^m{}_n$ to be the Levi-Civita connection of e^m_μ. On the other hand, the equation relative to θ^{mu}_μ produces the Einstein equations for the background e^m_μ. Thus, if e^m_μ is a solution of Einstein's equations in vacuum, then (18) together with the assumption of vanishing fermion fields yields a solution of the equations of motion of this theory.

One can understand better the dynamics of the gauge fields by inserting the VEV (18) in the action and neglecting interaction terms. The generalized actions (25) and (26) become

$$\mathcal{S}_{R1} + \mathcal{S}_\Theta \to \int d^4x\sqrt{g}\left[\frac{g_1}{16\pi}M^2 R + 4a_1 M^2\left(\Theta^m_{\mu\nu}\Theta^{\mu\nu}_m + 10K^j_\mu K^\mu_j\right)\right]. \tag{28}$$

Thus, one should identify the Planck mass as $M^2_{PL} = g_1 M^2$. Then, this shows that the isoboost gauge fields K^j_μ acquire mass at the Planck scale. As discussed in the introduction, also the spin-connection ω^j_μ, which is contained in $\Theta^m_{\mu\nu}$ and R, becomes massive. This can be seen most clearly for the constant background $e^m_\mu = \delta^m_\mu$; in curved backgrounds, it will generate masses for the fluctuations of ω around the Levi-Civita connection of e^m_μ. The W boson drops out of both terms because it commutes with the VEV and thus it remains massless.

Next, one can introduce an action quadratic in gravi-weak curvature:

$$\mathcal{S}_{R2} = \frac{1}{g_2^2}\int\left[r^{\bar{a}a\bar{b}b}_{\bar{e}e\bar{f}f}R^{\bar{e}e\bar{f}f} + (r^2)\theta^{\bar{a}a}\wedge\theta^{\bar{b}b}\right]\wedge\theta^{\bar{c}c}\wedge\theta^{\bar{d}d}\epsilon_{(\bar{a}a)(\bar{b}b)(\bar{c}c)(\bar{d}d)}. \tag{29}$$

This modifies the equations for the VEV, but flat space is still a solution. Inserting the VEV (18) and eliminating the $r^{\bar{a}a\bar{b}b}_{\bar{e}e\bar{f}f}$ auxiliary fields, this action reduces to a term quadratic in the gravitational curvature plus the standard Yang–Mills actions for the weak gauge fields:

$$\mathcal{S}_{R2} \to \frac{1}{g_2^2}\int d^4x\sqrt{g}\left(-R^j_{\mu\nu}R^{\mu\nu}_j - W^j_{\mu\nu}W^{\mu\nu}_j - K^j_{\mu\nu}K^{\mu\nu}_j\right). \tag{30}$$

Above the breaking scale, the gravi-weak symmetry manifests itself in the equality of the coefficients of all the three terms, while below the Planck scale the isoboosts and the spin connection are massive and decoupled. Due to the vanishing torsion, the $R^j_{\mu\nu}R^{\mu\nu}_j$ term contains higher derivatives for the graviton; its effect is however negligible relative to the Hilbert term at our energies: the phenomenological limits on its strength are very loose [11].

One should point out that, as in the standard Palatini gravity, the equations also admit the solution $\langle\theta\rangle = 0$. This corresponds to an 'unbroken' phase in which there is no distinction between gravitational and weak interactions. Since the metric is quadratic in θ, one expects this symmetric phase to be also 'topological'. The dynamical mechanism which favors the phase with nondegenerate metric is not well understood and will not be discussed here. Some attempts along these lines were made in [1, 7, 8].

In the broken phase, the comparison between the strength of gravitational and weak interactions has to be based on the effective Newton constant k^2/g_1M^2, where k is the energy scale. At the current available energies, this effective coupling is extremely small, while near the Planck scale it becomes comparable to the other couplings. Since all the other gauge interactions seem to converge to order-one couplings at a Grand Unification scale quite near the Planck scale, this can be taken as a hint toward the complete unification of all gauge and gravitational interactions. The present model where the weak and gravitational interactions are treated on equal footing is a step toward this direction.

Let us discuss the low-energy spectrum of this theory. The gravi-weak fermions ${\psi_L}^a$ contain the SM fermions and remain massless in this chiral world. In a nonchiral world, they should receive mass at the scale of electroweak breaking. As discussed above, among the six complex gravi-weak gauge fields the three 'isoboosts' and the six spin connection fields correspond to broken generators and acquire a Planck mass. Only the three W gauge fields remain massless in the broken phase. They should become massive at the lower energy scale of $SU(2)_L$ breaking.

The generalized soldering field $\theta_\mu^{\bar{a}a}$ gives rise to interesting structure in the broken phase. The full field has 64 real components that can be decomposed as follows:

$$\theta_\mu = Me_\mu^m(\hat{\sigma}_m \otimes \mathbf{1}_2) + h_\mu^{m4}(\hat{\sigma}_m \otimes \mathbf{1}_2) + \Delta_\mu^{mj}(\hat{\sigma}_m \otimes \sigma_j). \tag{31}$$

The first term on the rhs is the background, and h and Δ are fluctuations. These can be rewritten, using the VEV e_μ^m, as

$$h_{\mu\nu} = e_\nu^n h_\mu^n, \qquad \tilde{\Delta}_{\mu\nu}^j = e_\nu^n \Delta_\mu^{nj}. \tag{32}$$

The fluctuations around the background consist thus of the 16-components field $h_{\mu\nu}$, and three new (16 component) tensor fields $\tilde{\Delta}_{\mu\nu}^j$, one for each value of the $SU(2)_L$ index j. Since nine generators of the gravi-weak group are broken (corresponding to spin, boosts and isoboosts), nine of these fields can be fixed by the choice of unitary gauge. The natural choices are the six antisymmetric components of $h_{\mu\nu}$ and the three traces $\tilde{\Delta}_\mu^{\mu j}$. In this gauge the remaining degrees of freedom are the (ten) symmetric components of h, which after proper treatment of the diffeomorphism invariance become the physical graviton, and the (3×15) traceless components of $\tilde{\Delta}_{\mu\nu}^j$. These latter fields can also be decomposed in the antisymmetric (six components) and symmetric traceless (nine components) fields, that are copies of a (traceless) graviton. The emergence of these copies of tensor fields was noted in [12] where the gauge group of gravity was extended to a generic $SL(2N, C)$. In agreement with that analysis, one can see that the antisymmetric component of $\tilde{\Delta}$ does not get a kinetic term from the generalized EH term (25).

The symmetric part of $\tilde{\Delta}_{\mu\nu}^j$ represents new tensor particles that constitute an $SU(2)$ isospin triplet, therefore they have standard weak interactions and below the electroweak breaking scale they will consist of one neutral and two charged components. However all the direct couplings of $\tilde{\Delta}_{\mu\nu}$ with matter are Planck-suppressed, as for the graviton, because the common kinetic term is normalized with M^2.

Since these charged spin two particles are not observed at low energy, one can suppose that they have escaped detection because their mass is above the electroweak scale. While their mass could only be predicted in a complete model, some useful observations can still be made. First, one can write a $SU(2)_L$ gauge-invariant mass like $\mathrm{tr}(\tilde{\Delta}^j\tilde{\Delta}^j)$, which at first sight could be taken as large as the Planck mass. However, such a mass is actually the part of the expansion of the cosmological term around the background: [5]

$$\lambda\int\theta^{\bar{a}a}\wedge\theta^{\bar{b}b}\wedge\theta^{\bar{c}c}\wedge\theta^{\bar{d}d}\epsilon_{(\bar{a}a)(\bar{b}b)(\bar{c}c)(\bar{d}d)}=\lambda M^4\int \mathrm{d}^4x|e|+\lambda M^2\int \mathrm{d}^4x|e|\mathrm{tr}(\tilde{\Delta}^j\tilde{\Delta}^j)+\cdots. \quad (33)$$

It is therefore observationally constrained to be very small, because it is connected with the cosmological constant.

On the other hand, a different mass term may arise from the coupling with a Higgs field, that would give mass to $\tilde{\Delta}$ but not to the graviton, as was described in [12]. By this argument one may expect $\tilde{\Delta}$ to have mass in the weak range. It is then interesting to note that if its mass were slightly above the weak scale, for example $m_{\tilde{\Delta}}=300$ GeV, this particle would probably have escaped detection at LEP, being too heavy to be produced (in pair) and having a small decay rate (mainly through Higgs bosons). It should nevertheless be produced at LHC by the standard Drell–Yan gauge interactions $qq\to W\to\tilde{\Delta}\tilde{\Delta}$ at energy above $2m_{\tilde{\Delta}}$, and provide a nice signal of this theory. It is also interesting to speculate that depending on the model the lightest component of this triplet (usually the neutral one) may be stable and only weakly interacting, therefore being a candidate for dark matter. It would be interesting to carry out such an analysis in a complete model including the electroweak breaking sector.

5. Models with both chiralities

Even though in the SM left- and right-handed fermions occur in different representations of the gauge group, there are many unified models where at a more basic level the symmetry between left and right is restored. The minimal such models were based on the left–right symmetry [14, 15] and the Pati–Salam partial unification $SU(2)_L\times SU(2)_R\times SU(4)$ [16]. In these models, the hypercharge $U(1)$ group is enlarged to a group $SU(2)_R$ acting on the right-handed fermions in the same way as the weak $SU(2)_L$ acts on the left-handed ones. Then from the point of view of the Lorentz and weak groups, the fermions occur in the representations $(\mathbf{2},\mathbf{2})$ and $(\bar{\mathbf{2}},\mathbf{2})$ of $SL(2,\mathbb{C})\times SU(2)_L$ and $SL(2,\mathbb{C})\times SU(2)_R$, respectively (here we label representations by their dimension).

This suggests a first possible model, where we take two of the toy models considered in the previous section, with opposite chiralities, and join them to construct a semirealistic model of gravi-weak unification where fermions can have masses. Because infinitesimal Lorentz transformations are identified in one case with M^+_j generators and in the other case with M^-_j, we have to assume independent Lorentz groups for the two chiralities. Thus we start from a group $SO(4,\mathbb{C})_L\times SO(4,\mathbb{C})_R$, where the left gravi-weak group $SO(4,\mathbb{C})_L$ contains the left-Lorentz group $SL(2,\mathbb{C})_L$ and the weak gauge group $SU(2)_L$, while the right-gravi-weak group $SO(4,\mathbb{C})_R$ contains the right-Lorentz group $SL(2,\mathbb{C})_R$ and the internal gauge group $SU(2)_R$. In the low-energy broken phase, the physical Lorentz group will have to be identified with the diagonal subgroup of $SL(2,\mathbb{C})_L\times SL(2,\mathbb{C})_R$.

In this model, every massive fermion is realized by means of two fields Ψ_L,Ψ_R: a complex $SO(4,\mathbb{C})_L$-vector ($SO(4,\mathbb{C})_R$-singlet) and a $SO(4,\mathbb{C})_R$-vector ($SO(4,\mathbb{C})_L$-singlet). When decomposed into representations of their Lorentz and internal subgroups, they become the desired left-handed doublet of $SU(2)_L$ plus right-handed doublet of $SU(2)_R$.

The breaking of the two $SO(4,\mathbb{C})_{L,R}$ groups to the respective $SU(2)_{L,R}$ subgroups follows the scheme described in the previous sections, using separate generalized soldering forms $\theta^{\bar{a}a}_{L\mu}$

[5] Since $\tilde{\Delta}^j$ are traceless, this mass term is equivalent to the standard Pauli–Fierz mass for spin-two fields [13].

and $\theta^{\bar{b}b}_{R\mu}$. We assume that both VEVs have the form (18), so that they define the same background metric. Then, since the diffeomorphism group is unique, equation (20) written separately for the left and right transformations, will define a single residual global $SO(3,1)$ Lorentz group.[6]

The presence of two generalized soldering forms θ_L and θ_R leads also to two graviton fields. One of these is the standard massless graviton that is massless, thanks to diffeomorphism invariance, the other graviton may be naturally massive as discussed, e.g., in [17–19]. Likewise, also the fields $\tilde{\Delta}$ will be doubled, and thus one has two triplet tensor fields $\tilde{\Delta}_L$ and $\tilde{\Delta}_R$ of the left and right isospin, respectively.

The appearance of two independent gravi-weak groups may be somewhat unpleasant. It would be more elegant to have from the outset a single copy of $SL(2,\mathbb{C})$ to be identified with the Lorentz transformations. Because the representations are complex, we can think of the real Lorentz group $SL(2,\mathbb{C})$ as the complexification of the 'spin' group $SU(2)_S$ generated by M^+_j. Then, we must look for representations of $SU(2)_S \times SU(2)_L \times SU(2)_R$. The left-handed fermions are in the representation $(\mathbf{2},\mathbf{2},\mathbf{1})$ while the right-handed fermions are in $(\bar{\mathbf{2}},\mathbf{1},\mathbf{2})$. The (chiral spinor) representation $\mathbf{8}$ of $SO(7)$ decomposes precisely into $(\mathbf{2},\mathbf{2},\mathbf{1}) \oplus (\mathbf{2},\mathbf{1},\mathbf{2})$. Thus, a massive lepton doublet can be neatly accommodated into a single spinor representation of $SO(7,\mathbb{C})$, by including both the left and right (conjugated) fermions in the same multiplet.

In this model, the generalized soldering form needed to write the fermionic action will have two spinor indices in the $\mathbf{8}$ instead of two vector indices: $\theta^{\alpha'\alpha}_\mu$. Since in $SO(7,\mathbb{C})$ one has $\bar{\mathbf{8}} \times \mathbf{8} = \mathbf{64}_{\text{herm}} \oplus \mathbf{64}_{\text{antiherm}}$, one can take an Hermitian soldering form consisting of 64 real fields (for each spacetime index μ). Similarly to the simple chiral world described above, its VEV should be $e^m_\mu(\hat{\sigma}_m \otimes \mathbf{1}_2 \otimes \mathbf{1}_2)$, that leaves unbroken the three correct symmetries: the global Lorentz group plus the two local gauge groups $SU(2)_L$, $SU(2)_R$. The fluctuations of $\theta^{\alpha'\alpha}_\mu$ around this VEV contains 220 fields: the graviton $h_{\mu\nu}$ (10 components), two traceless tensor fields that are triplets under the two isospin groups, $\tilde{\Delta}^j_{\mu\nu L}$, $\tilde{\Delta}^j_{\mu\nu R}$ (45 components each) and a similar tensor field that is triplet under both left and right isospin groups $\tilde{\Delta}^{j_L j_R}_{\mu\nu}$ (120 components). All these new tensor particles, charged under $SU(2)_L$ or $SU(2)_R$, should take mass at the breaking of these symmetries.

Each of the two models described above has advantages and disadvantages. The model based on $SO(4,\mathbb{C})_L \times SO(4,\mathbb{C})_R$ has only 12 complex generators (whereas the model based on $SO(7,\mathbb{C})$ has 21 complex generators), and is therefore the minimal model that treats gravity and the weak force in a symmetric fashion. However, the gauge group is not simple. The model based on $SO(7,\mathbb{C})$ has a simple gauge group and each fermion doublet forms an irreducible representation. Although it is considerably larger than the previous group, it is the minimal simple group that contains both the Lorentz transformations and the left and right isospin groups such that weak doublets are in one irreducible representation.

We conclude this section by commenting on the possible origin of the electroweak breaking. While this part of the analysis should be carried on in a specific model including the hypercharge and the strong interactions, we find it useful to sketch the various possibilities. As is well known, the Higgs fields of the SM is an isospin doublet, Lorentz singlet. Since this is not a $\mathbb{Z}_2$-symmetric representation, the Higgs should be accompanied by the corresponding partner or embedded in a suitable larger representation.

From the point of view of group theory, the simplest possibility would be the analog of a Dirac spinor. This is the sum of a ψ^A field, that is a doublet of Lorentz, i.e. a Weyl spinor, and a ϕ^α field that is a doublet of isolorentz, i.e. an isospin doublet. The latter could play the role

[6] With different left and right VEVs this would generally not be the case, and the Lorentz invariance may be broken as in [19].

of the Higgs, and its VEV would lead to the correct breaking of isospin, without breaking the Lorentz invariance. The VEV would effectively break the $\mathbb{Z}_2$ symmetry, and may follow from a mechanism similar to the one leading to spontaneous breaking of LR-parity [15]. It is not clear however how a symmetry between a scalar doublet and a spinor singlet could be made consistent with the spin–statistics relation.

The $SU(2)_L$ symmetry could of course be broken also by the VEV of other fields. One example is a 'gravi-weak adjoint' representation Φ^a_b, with VEV in the isolorentz sector only: $\bar\Phi^{(-)j}(\mathbf{1}_2 \otimes \sigma_j)$. This is equivalent to a 'triplet breaking'. A similar breaking could be due to an additional VEV of the soldering form, in the $\tilde\Delta^j_{\mu\nu}$ components (see (31)). In general such a VEV would break the Lorentz symmetry: for example, $\tilde\Delta^j_{00}$ breaks the boosts transformations that may be marginally allowed, while $\tilde\Delta^j_{02}$ would break the rotations which is undesirable. A special possibility would be if $\tilde\Delta^j_{\mu\nu}$ were proportional to the background metric $g_{\mu\nu}$ for all j, i.e. $\langle\tilde\Delta^j_{\mu\nu}\rangle = g_{\mu\nu}\bar\phi^j$, that amounts to have a VEV for the traces of $\tilde\Delta^j$. In this case, it would not break the Lorentz and would behave just as the scalar triplet Φ^a_b just described.

A more promising possibility would be to introduce, in one of the models presented above, a field transforming under both the left and right $SU(2)$ weak groups, that would contain two left isospin doublets [14, 15]. Since the full electroweak breaking is intimately linked also to the breaking of $B - L$ or unified color groups, we shall defer a more complete analysis to a future work, and turn to the inclusion of strong interactions.

6. Including strong interactions

The fermion quantum numbers strongly suggest that the color $SU(3)$ group, together with the $U(1)$ group generated by $B - L$, forms the real group $SU(4) \approx SO(6)$ [16]. Let us then discuss briefly the inclusion of this extended color group in the models discussed above.

In the $SO(4,\mathbb{C})_L \times SO(4,\mathbb{C})_R$ case, it is natural to try to unify first the two groups in $SO(8,\mathbb{C})$. Indeed, the two spinors $(\mathbf{4},\mathbf{1}) \oplus (\mathbf{1},\mathbf{4})$ can be obtained from the reduction of the $\mathbf{8}$ (vector) of $SO(8,\mathbb{C})$. Then, the fermions of one SM family should be in the $\mathbf{8}$ (vector) of $SO(8,\mathbb{C})$ and in the $\mathbf{4}$ (chiral spinor) of $SO(6) \approx SU(4)$. Unfortunately with orthogonal groups it is always impossible to obtain a product of the vector and spinor representations from the reduction of irreducible representations of a larger group, and thus it seems that having decided in the first place to introduce the gravi-weak orthogonal group $SO(4,\mathbb{C})$ we have precluded the possibility of unifying it with the strong interactions in a simple group[7].

Instead, the $SO(4,\mathbb{C})_L \times SO(4,\mathbb{C})_R$ model points in the direction of the framework described in [20] where, using Clifford algebras, this unification is achieved in a different and geometrical way. There the $SU(4)$ groups are also duplicated as $SU(4)_L \times SU(4)_R$ and, in the symmetric phase, the left and right sectors have completely independent degrees of freedom, both in the gauge and in the gravitational sectors. In the broken phase, as the gravi-weak groups are broken and reduced to the single Lorentz group, also the two color groups should be broken to a diagonal $SU(4)$ and then to the color $SU(3)$.

The approach based on $SO(7,\mathbb{C})$ allows the complete unification in a simple group but suffers from another problem. Here a SM family is contained in the representation $(\mathbf{8},\mathbf{4})$ of $SO(7,\mathbb{C}) \times SO(6,\mathbb{C})$, and under the inclusion $SO(7,\mathbb{C}) \times SO(6,\mathbb{C}) \subset SO(13,\mathbb{C})$, one can indeed obtain the $(\mathbf{8},\mathbf{4})$ from the reduction of the (spinor) $\mathbf{64} \to (\mathbf{8},\mathbf{4}) \oplus (\bar{\mathbf{8}},\bar{\mathbf{4}})$. The breaking $SO(7,\mathbb{C}) \to SL(2,\mathbb{C}) \times SU(2)_L \times SU(2)_R$, leads to the further decomposition $(\mathbf{8},\mathbf{4}) \to (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{4}) \oplus (\bar{\mathbf{2}},\mathbf{1},\mathbf{2},\mathbf{4})$, showing explicitly that this is exactly a family of the SM. However, the decomposition of the $\mathbf{64}$ contains also a so-called mirror family,

[7] The situation may change by looking for embeddings in non-orthogonal groups.

$(\bar{\mathbf{8}}, \bar{\mathbf{4}}) \rightarrow (\bar{\mathbf{2}}, \mathbf{1}, \mathbf{2}, \bar{\mathbf{4}}) \oplus (\mathbf{2}, \mathbf{2}, \mathbf{1}, \bar{\mathbf{4}})$, that should be disposed off somehow. The problem of eliminating mirror fermions has no simple solution, mainly because one cannot give a gauge-invariant mass to the undesired chiral $SU(2)$ doublets [21]: their mass would break the weak symmetry and thus it cannot be higher than the electroweak scale.

This approach is very close in spirit to the original $SO(13, 1)$ unified theory [1], where the chiral spinor $(\mathbf{64})$ of $SO(13, 1)$ decomposes under $SO(3, 1) \times SO(10)$ into $(\mathbf{2}, \mathbf{16}) \oplus (\bar{\mathbf{2}}, \overline{\mathbf{16}})$, and the $(\bar{\mathbf{2}}, \overline{\mathbf{16}})$ representation consists of Weyl mirror fermions.

7. Avoiding Coleman–Mandula

The Coleman–Mandula theorem [3] states that under certain natural hypotheses (which are very likely to hold in the real world) the symmetry group of the S-matrix must be a direct product of the Lorentz group and an internal symmetry. It is usually interpreted as a no-go theorem, forbidding a nontrivial mixing of spacetime and internal symmetries. Supersymmetry and certain quantum groups famously manage to avoid the theorem: in these cases the symmetry is not an ordinary Lie group, as assumed by the theorem. The proposal for unification discussed in the previous sections is based on ordinary Lie groups and thus superficially may seem to violate the theorem. We will discuss here why this is not the case.

To explain this point, we have to discuss first the fate of the Lorentz group in the proposed unified models. It is important to distinguish the local Lorentz transformations acting on the internal spaces, which are gauge transformations and are present independently of the background, from the global Lorentz transformations which are only defined as the subgroup of diffeomorphisms $x'(x)$ that leave the background (Minkowski) metric invariant. We can then address the question whether these transformations are broken or not. The 'unitary gauge' choice (18), with $e_\mu^m = \delta_\mu^m$ breaks both the $SL(2, \mathbb{C})_+$ local Lorentz and these global Lorentz transformations. However, as discussed in section 3, the VEV is invariant when a global Lorentz transformation Λ is compensated by an internal transformation with parameter $S = \mathcal{D}^{(\frac{1}{2},0)}(\Lambda)$. It is this global Lorentz symmetry group that enters into a discussion of the Coleman–Mandula theorem.

One of the hypotheses of the Coleman–Mandula theorem is the existence of a Minkowski metric. Thus, from the point of view of the theory described above, this means that it can only apply to the 'broken' phase, more precisely to the special case when the ground state is flat space. But we have shown that in the broken phase the residual symmetries are precisely a global Lorentz symmetry and a local internal symmetry. This is in complete agreement with the Coleman–Mandula theorem.

The greater symmetry of the unified theory would only manifest itself in the situation where the soldering form vanishes, which would correspond to a symmetric 'topological' phase of the theory; in that phase there would be no metric on spacetime, let alone a Minkowski metric, and the hypotheses of the Coleman–Mandula theorem would not apply. This argument applies also to other theories such as the one discussed in [1].

The main lesson to be drawn from this discussion is therefore that there need not be a contradiction between the Coleman–Mandula theorem and theories that mix internal and spacetime transformations: the theorem does not forbid such a nontrivial mixing, as long as it manifests itself only in a phase with no metric.

8. Discussion and conclusions

There have been many attempts at unifying gravity with the other interactions [22]. The one we discussed here generalizes in the most straightforward way the philosophy and the procedures that are believed to work, in particle physics, for the other interactions. We have considered mainly the symmetry between the (left or right) weak interactions and the gravitational ones, in a 'gravi-weak' group $SO(4, \mathbb{C})$. The unification of spin and isospin transformations is very natural in view of the quantum numbers of the fermion fields[8]. Since the group $SO(4, \mathbb{C})$ is not simple, the gauge fields associated to its commuting factors $SL(2, \mathbb{C})_+$ and $SL(2, \mathbb{C})_-$ could in general have different couplings, and in this sense the two interactions would not be truly unified. In analogy to what is done in left–right-symmetric models, we have postulated the existence of a discrete $\mathbb{Z}_2$ symmetry exchanging the two sectors, that would force this unification. This symmetry can be seen as a remnant of the unification in a larger simple group, as discussed in sections 5 and 6. Therefore, we have a unification of the gravitational and weak interactions in the sense described in the introduction.

The gravi-weak unification requires a generalized vierbein or soldering form that naturally acts as an order parameter. Its VEV defines the gravitational background and at the same time selects the weak isospin group as the only unbroken gauge group. By using the soldering form as an order parameter, we also avoid the potential obstruction provided by the Coleman–Mandula theorem. The conditions of the theorem, in particular the existence of a metric, are only satisfied in the broken phase, and indeed in that regime the model predicts that the symmetry of the theory is the product of spacetime and internal symmetries.

After the symmetry breaking, the extended soldering form gives rise, in addition to the standard graviton, also to an isospin triplet traceless tensor field. The gauge-invariant mass of this spin-two triplet is connected with the cosmological constant and thus is constrained to be small. On the other hand, its mass may arise from the mechanism of electroweak breaking, leading to the interesting possibility that this particle, while having escaped detection up to now, may be directly observed at LHC.

We have then discussed extensions of this scheme to include massive fermions, and briefly analyzed also the inclusion of the strong interactions. At the moment there seem to be (at least) two possible frameworks.

One is based on the use of $SO(7, \mathbb{C})$ that unifies in a simple group both the left and right isospin groups with the Lorentz group. This is the minimal simple group that can accommodate a massive fermion in a single irreducible representation. After inclusion of the strong interactions, this approach however suffers from the well-known problem of mirror fermions, because only vector-like fermions are generated and the undesired chiral copy cannot be given mass higher than the weak scale [21]. The situation is thus similar to the one encountered in the approach originally proposed in [1], where all the interactions are unified in a group $SO(13, 1)$ and gravity is separated by the VEV of a soldering form. There the multiplets of the resulting $SO(10)$ gauge interactions always appear in vector-like couples $\mathbf{16}+\overline{\mathbf{16}}$.

A second scheme, closer in spirit to left–right theories, postulates the duplication of the gravi-weak group in left and right copies at the most fundamental level, $SO(4, \mathbb{C})_L \times SO(4, \mathbb{C})_R$. This approach does not suffer from the problem of mirror fermions, but calls for such a duplication also in the strong sector, $SU(4)_L \times SU(4)_R$. These duplications point toward a more geometric framework, discussed in a separate work [20], that is based on the use of Clifford algebras.

To conclude, the bottom-up approach that motivated the present work has led to the result that one can successfully treat the weak and gravitational interactions on equal footing. It also showed possible ways to construct scenarios of complete unification including the strong

[8] It is well known that spin and isospin degrees of freedom can mix in solitonic solutions [23]. In this work, we have suggested that such mixing may occur at the level of fundamental degrees of freedom.

interactions. The problems that arise are similar to the typical ones that occur in grand unified theories; on the other hand, contrary to common belief, there seems to be no fundamental obstacle to the unification of gravity with other gauge interactions using the familiar methods of particle physics. We think that further exploration of these scenarios will provide novel insight both at the fundamental and phenomenological level.

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